9-2002

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SMU ECONOMICS & STATISTICS
WORKING PAPER SERIES

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September 2002

Paper No. 16-2002

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28 September, 2002

Abstract

This paper examines the allocative efficiency of two popular non-price allocation mechanisms – the lottery (random allocation) and the waiting-line auction (queue system) – for the cases where consumers possess identical time costs (the homogeneous case), and where time costs are correlated with time valuations (the heterogeneous case). We show that the relative efficiency of the two mechanisms depends critically on the scarcity factor (measured by the ratio of the number of objects available for allocation over the number of participants) and on the shape of the distribution of valuations. We obtain a set of analytical results showing that the lottery generally dominates the waiting-line auction unless the there are very few high-valuation individuals and the scarcity factor is sufficiently high. We further demonstrate that while consumer heterogeneity may improve the relative allocative efficiency of the waiting-line auction, this is usually not significant enough to reverse the general dominance of the lottery.

KEY WORDS: Lottery; Non-price allocation, Rent-seeking; Waiting-line auction

JEL CLASSIFICATION: C15; D44, D61

*This is a revised version of the Working Paper #16-2002, School of Economics and Social Sciences, Singapore Management University. We thank Shmuel Nitzan, Raaj Sah and our colleagues at the Singapore Management University for their comments and suggestions that have improved the exposition of this paper.

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Abstract

This paper examines the allocative efficiency of two popular non-price allocation mechanisms – the lottery (random allocation) and the waiting-line auction (queue system) – for the cases where consumers possess identical time costs (the homogeneous case), and where time costs are correlated with time valuations (the heterogeneous case). We show that the relative efficiency of the two mechanisms depends critically on the scarcity factor (measured by the ratio of the number of objects available for allocation over the number of participants) and on the shape of the distribution of valuations. We obtain a set of analytical results showing that the lottery generally dominates the waiting-line auction unless the there are very few high-valuation individuals and the scarcity factor is sufficiently high. We further demonstrate that while consumer heterogeneity may improve the relative allocative efficiency of the waiting-line auction, this is usually not significant enough to reverse the general dominance of the lottery.

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1 Introduction

Singapore celebrates its National Day on 9 August, and every year, about 60,000 tickets for the National Day Celebrations – which include a parade, musical performances and fireworks – are distributed free, on a first-come, first-served basis. As the date and venues of the distribution were announced in advance, this led to queues forming days before the distribution date as eager Singaporeans took leave from work to wait in line.[1] Many Singaporeans felt that the queuing system was inequitable and inefficient, and a lottery system would be the fairest and most efficient way to allocate the tickets. After all, the allocation of tickets by lottery to attend mass events is widely practiced around the world. For instance, in the United States, a portion of the tickets to attend each game of the SuperBowl series of American football is distributed via a lottery among season-ticket holders of the two competing teams. Others have suggested that auctioning the tickets would be a better system as the tickets will go to individuals who value them most, and the proceeds could be channeled for charitable purposes.[2] The National Day tickets are highly-prized; an active market for National Day tickets exists, and tickets were being offered for sale on the Internet, at up to 500 Singapore dollars per ticket!

What is the appropriate mechanism for allocating the National Day tickets: a lottery, a queue or a market auction? Under what circumstances will one allocation mechanism be preferred over another mechanism?[3] The choice of an optimal non-price allocation mechanism is significant in a wide variety of economic contexts, such as the allocation of subsidized public housing or the award of oil drilling leases. In many countries, the provision of public goods and services is often subject to price and quantity controls, which, in turn, necessitates the use of non-price mechanisms to allocate the scarce resources.

In our introductory example, a market auction of the National Day tickets is potentially lucrative, as the potential revenue is likely to substantially higher than the administrative cost of holding the auction. However, commercialization of the National Day celebrations is politically unacceptable. Theoretically, if administrative cost is negligible, a market auction offers the most efficient allocation and the auction revenue can be re-distributed, so that social welfare is preserved and possibly enhanced. Thus, under ideal circumstances, there need not be a tradeoff between efficiency and equity. In general, when the value of the

[1] It was reported in the press that some individuals had queued for as long as 35 hours to obtain a pair of tickets. See "Cut out the queues, ballot for National Day Parade tickets,"*The Straits Times* July 9, 2002.

[2] Singapore already has the distinction of being the only country in the world to hold a monthly auction of car licenses that allow car owners to keep their cars for 10 years. Since the vehicle quota system was introduced in April 1990, the Singapore government receives proceeds of approximately one billion Singapore dollars annually from the car license auctions as contribution to its Consolidated Fund.

[3] There have also been suggestions to allocate a portion of the tickets by merit, according to a set of criteria measuring an individual's contribution to community service.

objects to be allocated are significant (either private utilities or potential profits), so that allocative efficiency is important, a market auction is the optimal allocation mechanism. Thus, car licenses, residential land plots and band spectrum are allocated through market auctions. If an auction is not feasible, due to either economic or political reasons, two commonly used non-price mechanisms are the lottery and the waiting-line auction. A lottery is a random allocation to interested individuals while a first-come first-served queue system is akin to a waiting-line auction, where participants compete by choosing the optimal time to arrive at the queue, which then determines their expected probabilities of getting an allocation.

There are many examples of resources being allocated through a lottery system; these include hunting permits, immigrant visas, admission to schools and universities, rights to fishing berths and oil drilling leases. In some countries, economic burdens, such as military draft and jury duty are also decided by lottery.[4] Publicly provided resources that have been allocated through a queue system include subsidized medical care, recreational facilities and public housing. In some instances, participation in the waiting-line auction or in the lottery is conditional on a pre-selection based on *merit* – i.e. the participants must meet certain criteria in order to be eligible to compete.

There are two aspects of an allocation mechanism to consider: *allocative efficiency* and *equity*. The *efficiency* of a mechanism is measured by the degree of rent dissipation, which comprises *resource misallocation* and *rent seeking costs*. Resource misallocation occurs when individuals who succeed in obtaining an allocation are not be the ones who value the objects most. In the case of the lottery, the likelihood of a correct allocation is inversely proportional to the number of participants, while in the case of the waiting-line auction, the individuals who queued up earlier may be the ones with lower opportunity cost of time rather than the ones with higher monetary valuations. The other source of rent dissipation is rent-seeking costs. For instance, waiting in line creates disutility as well as potential loss in income. If there is pre-selection for participation in a lottery, or if the probability of allocation can be influenced, rent-seeking behavior may emerge even in a lottery allocation scheme.

The *equity* of an allocation mechanism is measured by the welfare impact of the allocation outcome, according to a social welfare function. The choice of one allocation mechanism over another often necessitates a tradeoff between efficiency and equity.[5] The preference for the lottery to allocate economic goods (and burdens) is frequently made on the grounds of horizontal equity, i.e. individuals who possess the same relevant characteristics or qualities should be treated equally. In the context of optimal taxation policy, Stiglitz (1982) has

[4]One could perhaps attempt to avoid being allocated an economic *burden*; for instance, by disqualifying oneself through a medical condition in the case of a military draft, or making oneself unsuitable for jury duty.

[5]See Elster (1991) for a discussion of this issue.

shown that a small degree of randomization in tax rates can improve welfare. If transferability is feasible and not prohibited, a lottery can potentially attain first-best allocative efficiency, albeit with a re-distribution of social surplus to lucky winners in the lottery. However, this may give rise to potential rent-seeking behavior, thereby reducing the efficiency of the lottery. Boyce (1994) has argued that lottery participation fees and restrictions on transferability may improve the relative efficiency of the lottery, as these measures reduce rent-seeking behavior. The same argument can be made for rationing goods through a waiting-line auction, as transferability alters the incentives of participation and the optimal strategy, as noted in Sah (1987).

Although there is an extensive literature (see Section 1.1) on the properties of different non-price allocation mechanisms, relatively little work has been carried out to compare the allocative efficiency of the lottery versus the waiting-line auction. While it is recognized that the efficiency and desirability of the lottery versus the waiting-line auction depends on the specific circumstances under consideration, there is no general theoretical results, thus far, to delineate the circumstances under which one mechanism would dominate the other. Furthermore, it is also not clear which are the key factors determining the relative efficiency of the two mechanisms, and what is the impact on relative efficiency when valuations and time costs are correlated. In a related paper, Taylor, Tsui and Zhu (2001) studied this issue and provided a set of initial results on the relative efficiency of the two allocation mechanisms. Using numerical analysis, they also considered the local impact of mean-preserving dispersions in valuations on relative allocative efficiency. Our motivation for the present paper is to attempt a more general analysis of the relative allocative efficiency of the two mechanisms.

We investigate this problem for the cases where consumers possess identical time costs (the homogeneous case), and where time costs are correlated with time valuations (the heterogeneous case).Our approach is as follows. For the case where consumers possess identical time costs (the homogeneous case), we derive general analytical expressions for the expected social surplus functions under the assumption that the time valuation follows a power function distribution, a Weibull distribution, a logistic distribution, a beta distribution, etc. These distributions are chosen for our study as they cover all the potential situations of practical interest.[6] The set of analytical results allows us to draw the following general conclusions: (i) the shape of the distribution of time valuations and the scarcity

[6]The type of distributions covered in terms of the shape of their density functions include uniform (consumer valuations on the objects are equally distributed), L-shaped (majority of consumers with very low valuations and a few with very high valuations), U-shaped (most consumers have either very high or very low valuations, with a few in between), J-shaped (majority of consumers with high valuations and a few low valuations), unimodal or hump-shaped (most consumers have valuations in the middle, with a few having low or high valuations).

factor (measured by the ratio of number of objects over number of participants) are the two key factors in determining the relative efficiency of the two mechanisms, and (ii) the lottery generally dominates the waiting-line auction unless the distribution of time valuations is extremely skewed (L-shaped), so that there are relatively few participants possessing high valuations, and the scarcity factor is sufficiently high.

We extend the analysis to the case where consumers are heterogenous in their time costs. We study a model where time costs and time valuations are correlated, and show that the relative efficiency of the waiting-line auction improves when there is a positive correlation between time costs and time valuations, but deteriorates when the correlation is negative. However, the improvement in the allocative efficiency of the waiting-line auction is not sufficient to alter the overall dominance of the lottery. When time costs and valuations are independent, the results for the case of homogeneous time costs continue to hold. The generality of our results suggests that introducing an element of randomization in the allocation process may help to improve the allocative efficiency in a wide variety of rent-seeking situations.

The rest of the paper is organized as follows. We briefly review some of the related literature in Section 1.1. Section 2 presents the basic analytics of the two allocation mechanisms. Section 3 compares efficiencies of the two mechanisms for the case of homogenous consumers under various distributional assumptions for time valuations. Section 4 extends the analysis to the case of heterogenous consumers. Section 5 concludes the paper.

1.1 Related literature

One of the early papers to study the issue of allocative efficiency is Oi (1967), who investigated the economic cost of the military draft lottery due to the misallocation of resources. Several authors (Eckhoff (1989) and Goodwin (1992)) have examined the desirability of the lottery from an horizontal equity perspective, arguing that lotteries represent a fair means to allocate resources, as everyone has the same probability of winning, regardless of the characteristics or qualities that one possess. Boyce (1994) examined both the efficiency and equity of the lottery system; his results suggest that the lottery is a dominant allocation mechanism when the proportion of redistributable social surplus is small. An early survey on the theory of rationing was provided by Tobin (1952). The economics of the queue system in rationing goods was analyzed by Nicholas, Smolensky and Tideman (1971) and Barzel (1974). The issue of heterogeneity was taken up in Suen (1989), who analyzed the efficiency of the waiting-line allocation system when individuals differ in both time costs and personal valuations. Suen's analysis indicates that an increase in the degree of dispersion in time costs may lead to a reduction in rent dissipation. Furthermore, transferability and the introduction of a secondary market for a rationed good may not necessarily improve welfare as the total surplus under rationing by waiting depends more on the degree of dispersion than on the level of time costs and personal valuations.

An important paper on the subject of waiting-line auction was Holt and Sherman (1982), who formulated the waiting-line auction as a non-cooperative game with incomplete information. In their model, each participant's choice problem is to decide whether to join the queue, conditional on an expected waiting time. Under this formulation of the waiting-line auction, the optimal choice of the time of arrival at the queue, and thus the expected waiting time and probability of success, is strategically the same as the selection of an optimal bid in a sealed-bid tender.[7] The approach adopted by Holt and Sherman(1982) facilitates an analysis of the rent-seeking costs in the waiting-line auction.

2 The Model

There are m identical and indivisible objects to be distributed freely to a group of $n(> m)$ consumers, at most one object per person, using either the lottery or the waiting-line allocation rule. The opportunity costs of time of the n consumers (measured by their wage rates) are denoted by $w_1, w_2, \cdots, w_n$, and their monetary valuations of the objects (measured in dollars) are denoted by $v_1, v_2, \cdots, v_n$. Thus, the ratio $y_i = v_i/w_i$ describes an individual's valuation of an object measured in time units. We shall refer to v_i as *monetary valuation* and y_i as *time valuation* for the ith consumer.

Consumers are risk neutral. They all know m, n, their own time valuations and time costs. However, they cannot observe the time valuations and time costs of other participants. Each participant has identical subjective beliefs about the possible monetary valuations and time costs of the other consumers. Specifically, each person believes that the monetary valuations and time costs for the $n-1$ rival claimants are independent realizations of a pair of continuous random variables $\{V, W\}$ having a joint distribution function $F(v, w)$ with support $[\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}]$, for some finite non-negative $\underline{v}$ and positive $\underline{w}$. The marginal distributions of V and W are denoted by $F_V(v)$ and $F_W(w)$, respectively. Similarly, the marginal distribution of Y is denoted by $F_Y(y)$.

We compare the efficiencies of two allocation mechanisms – the lottery and the waiting-line auction – using as our measure of efficiency the *expected social surplus.* We define this as the sum of the expected payoffs for all n consumers. We first present the basic analytics of the two allocation mechanisms.

[7]see McAfee and McMillan (1987) for an introduction to the auction literature.

2.1 Lottery

The lottery mechanism operates as follows: at a pre-specified time, m consumers will be chosen randomly out of the n participants and each of them will be allocated one object. The probability that the ith customer obtains an object is identical and equal to

$$H^R = \frac{m}{n}.$$

Given the ith consumer's monetary valuation v_i, his monetary payoff is

$$\pi^R(v_i) = v_i H^R = \frac{m v_i}{n}.$$

Given the symmetric treatment of all consumers, the expected social surplus under the lottery mechanism is

$$S^R = nE[\pi^R(V)] = mE(V). \tag{1}$$

Hence, the expected social surplus under the lottery mechanism depends only on the number of objects available for allocation and the mean value of the distribution of monetary valuations.

2.2 Waiting-line auction

In a waiting-line auction or queue system, the objects are allocated at a pre-specified time and location, on a first-come-first-served basis. Each consumer may occupy only one position in the queue and only one object is allocated to each successful person in the queue. We assume that individuals who arrive after mth person in the queue will be notified so that no unsuccessful queuers will waste time in the queue.[8] We also assume that the time for each individual to reach the queue is negligible compared with the actual waiting time.

The equilibrium queuing time. For each individual, there is an optimal (equilibrium) queuing strategy $\tau(y_i), i = 1, \cdots, n$, which is a function of the individual's time valuation y_i. If the individual chooses the arrival time according to the equilibrium queuing strategy, i.e., $t_i = \tau(y_i)$, his expected payoff will be globally maximized (see Holt and Sherman(1982)). In equilibrium, individuals with higher valuations for the objects will choose optimally to arrive earlier at the queue and wait a longer period of time. This implies that the function $\tau(y)$ is a positive-valued and strictly increasing. Holt and Sherman (1982) further assumed that it is differentiable and provided a solution of the equilibrium queuing time function (their Equation (9)) under a slightly broader setting, which becomes, in our current setting

[8] As Holt and Sherman (1982) shows, if losers have to wait, each consumer will reduce their optimal waiting time, so that in equilibrium, the expected waiting time and payoff for each consumer remains unchanged. Hence, the results of this paper are not sensitive to this assumption.

$$\tau(y) = \frac{1}{H_Y^Q(y)} \int_{\underline{y}}^{y} x h_Y^Q(x) dx = y - \frac{1}{H_Y^Q(y)} \int_{\underline{y}}^{y} H_Y^Q(x) dx \tag{2}$$

where $h_Y^Q(y)$ and $H_Y^Q(y)$ are, respectively, the density function and the distribution function of the mth largest order-statistic among the $n-1$ independent draws from the population distribution of time valuations for the objects. Note that the second part of the equation follows from integration by parts. Recall that the marginal distribution of Y is denoted by $F_Y(y)$. It is straightforward to verify that

$$H_Y^Q(y) = \sum_{k=n-m}^{n-1} \binom{n-1}{k} [F_Y(y)]^k [1 - F_Y(y)]^{n-k-1}.$$

Note that the subscript '$_Y$' indicates that the distribution Y is used.

When all the n consumers choose their waiting time according to the equilibrium queuing strategy given in (2) above, the probability that the ith consumer will receive an allocation is simply $H_Y^Q(y_i)$ (using the fact that $\tau(y_i)$ is a strictly increasing function of y_i). Holding m and n constant, consumers who possess higher time valuations will choose a longer waiting time and therefore an earlier arrival in the queue.

The equilibrium expected payoff. The equilibrium expected payoff, in time units, for the ith consumer, is

$$\pi^Q(y_i) = (y_i - \tau(y_i)) H^Q(y_i) = \int_{\underline{y}}^{y_i} H_Y^Q(x) dx$$

which,upon multiplying the time cost w_i gives the equilibrium expected money payoff for the ith consumer. Finally the equilibrium expected social surplus generated by the waiting-line auction is

$$S^Q = nE\left(W \int_{\underline{y}}^{Y} H_Y^Q(x) dx\right). \tag{3}$$

Note that S^Q depends on the joint distribution of time valuation Y and time cost W. Thus, in order to compare the allocative efficiency of the two allocation mechanisms, we need to specify the joint distribution of Y and W. Once this is specified, we can derive theoretical closed-form expressions for S^R and S^Q, and compute the ratio S^R/S^Q.

3 Efficiency Comparison: Homogeneous Consumers

We first consider the case of the homogeneous consumers. By this, we shall refer to a situation where the m consumers belong to the same income class, so that the degree of variation in time costs is small relative to the monetary valuations. In such a case, it is reasonable to treat time costs as the same, i.e., $w_i = w_c, i = 1, \cdots, n$. With this specification, we can show that the expected social surplus generated by a waiting-line auction can be expressed solely in terms of the monetary valuation $V = Yw_c$, i.e.,

$$S^Q = nE\left(\int_{\underline{v}}^{V} H_V^Q(x)dx\right),$$

where,

$$H_V^Q(v) = \sum_{k=n-k}^{n-1} \left(\begin{array}{c} n-1 \\ k \end{array} \right) [F_V(v)]^k \, [1 - F_V(v)]^{n-k-1}$$

Therefore, S^Q is independent of the actual value of the constant w_c. Furthermore, by switching the order of integrations and then the order of integration and summation, we can derive

$$\begin{aligned} S^Q &= n\int_{\underline{v}}^{\overline{v}} \left(\int_{\underline{v}}^{v} H_V^Q(x)dx\right) f_V(v)dv \\ &= n\int_{\underline{v}}^{\overline{v}} \left(\int_{x}^{\overline{v}} f_V(v)dv\right) H_V^Q(x)dx \\ &= n\int_{\underline{v}}^{\overline{v}} [1 - F_V(v)]H_V^Q(v)dv \\ &= n\sum_{k=n-m}^{n-1} \left(\begin{array}{c} n-1 \\ k \end{array} \right) \int_{\underline{v}}^{\overline{v}} F_V(v)^k[1 - F_V(v)]^{n-k}dv \end{aligned} \tag{4}$$

Thus, S^Q depends on the number of the objects to be allocated m, the number of the consumers n, and the probability distribution of consumers' monetary valuations $F_V(v)$. The expression for S^R remains unchanged, whether consumers have identical or different time costs, as it depends only on the number of objects to be allocated m and the expected value of the consumers' monetary valuations $E(V)$.

In the following sub-sections, we present a set of general results on the relative allocative efficiency (S^R/S^Q or S^Q/S^R) of the two allocation mechanisms, under various distributional assumptions on the monetary valuation V. Common classes of distributions such as power function, Weibull, logistic and beta are considered. These four classes of distributions cover the broad range of distribution forms that the monetary valuation could reasonably possess[9], and thus allow us to examine the effect of the distributional shape on the relative efficiency of the two mechanisms. To facilitate the exposition of our analysis, we summarize the technical results into lemmas, and the qualitative conclusions into propositions (which are direct consequences of the results of the corresponding lemmas). The proofs of all lemmas are provided in the Appendix.

[9] The broad range of distributional forms include L-shaped, U-shaped, J-shaped, flat, unimodal, etc. Detailed explanations will be provided along the introduction of the results.

3.1 Monetary valuation with the power function distribution

We begin our analysis with the power function distribution, as this is the only specific distributional form, to our knowledge, that had been considered in the literature.[10] We present a set of general analytical results that allow for a clear-cut differentiation between the two mechanisms.

The cumulative distribution function of the power function distribution has the following form:

$$F(v;\theta,\beta)=\left(\frac{v}{\theta}\right)^{\beta},\ 0\le v\le\theta;\theta>0,\beta>0, \tag{5}$$

where β controls the shape of the distribution, called the *shape parameter*, henceforth. In particular, the probability density function (pdf) is decreasing (L-shaped) when $\beta<1$, constant (uniform) when $\beta=1$, and increasing (J-shaped) when $\beta>1$. See Figure 1 for a graphical illustration. The parameter θ controls the range or scale of the V values, and hence, called the *scale parameter.* The mean and variance of the distribution are $E(V)=\frac{\theta\beta}{\beta+1}$ and $Var(V)=\frac{\theta^2\beta}{(\beta+2)(\beta+1)^2}$, respectively.

Figure 1

Lemma 1: *If consumers' monetary valuations are drawn from the power function distribution, the expected social surplus functions associated with the two allocation mechanisms are* $S^R=m\theta\beta/(\beta+1)$ *and* $S^Q=S^R h(\beta,n,m)$*, respectively, where*

$$h(\beta,n,m)=\frac{n}{m}-\frac{n!\,(\beta n+m+1)\Gamma(n-m+\frac{1}{\beta})}{\beta m(n-m-1)!\,\Gamma(n+1+\frac{1}{\beta})}, \tag{6}$$

with $\Gamma(\cdot)$ *being the gamma function. Furthermore,* $h(\beta,n,m)$ *satisfies the following: (i) it is strictly increasing in* m*, decreasing in* n*, and decreasing in* β*; (ii)* $h(1,n,m)=\frac{m+1}{n+1}$*; (iii)* $h(\frac{1}{2},n,m)=\frac{3mn+3n+2-2m^2}{(n+1)(n+2)}$*; (iv)* $h(\beta,n,1)=\frac{n(1+\beta)}{(1+n\beta)(1+n\beta-\beta)}$*; and (v)* $\lim_{\beta\to\infty}h(\beta,n,m)=0$.

The $h(\beta,n,m)$ function in Lemma 1 completely characterizes the relative efficiency of the lottery versus the waiting-line auction. If the function $h(\beta,n,m)$ takes on values smaller (greater) than 1, this implies that the lottery mechanism dominates (is dominated by) the waiting-line auction. It is important to note that h depends on the monetary valuation distribution only through the shape parameter β. The larger the β, the smaller the h, and hence the larger is the S^R/S^Q. It is also important to note that h depends negatively on

[10] Taylor, et al. (2001) studied the case of the power function distribution, but limited their analysis to the case where $m=1$. They also considered the beta distribution with $\beta=1$ in their numerical analysis. However, as we shall note in Section 3.4, this is, in fact, also a special case of the power function distribution.

the scarcity factor m/n. Hence, the smaller the m/n, the larger the S^R/SQ. Lemma 1 leads immediately to the following proposition.

Proposition 1: *Suppose consumers' monetary valuations are drawn from the power function distribution. For any given θ, m and n, the lottery dominates the waiting-line auction when $\beta \geq 1$. Furthermore, the degree of dominance increases as β increases. When $\beta < 1$, the lottery continues to dominate the waiting-line auction if the ratio m/n is sufficiently small (For instance, we can show that $h(\frac{1}{2}, n, m) \leq 1$ if $m/n \leq \frac{1}{2}$).*

The intuition underlying Proposition 1, and for the other results in Section 3, is as follows. First, note that rent dissipation in the lottery occurs only as a result of resource misallocation. By Lemma 1, S^R is increasing in both m and β (since $E(V)$ in increasing in β). Since resource misallocation occurs when the m objects may be allocated to a set of consumers that do not value them the most, the extent of this inefficiency, and therefore the magnitude of rent dissipation, becomes smaller if there are more objects to be distributed in a lottery. This follows directly from the fact that the probability of a correct allocation improves as ratio m/n is increased.

By contrast, in waiting-line auction, there is no resource misallocation when time costs are homogeneous. This is because the optimal waiting time is monotonic in monetary valuation, so that a consumer with a higher valuation would queue up earlier. The source of the allocative inefficiency in the waiting-line auction is the rent-seeking cost of waiting in line. Thus, if the rent-seeking cost of the waiting-line auction is higher (lower) than the resource misallocation of the lottery, the waiting-line auction is the less (more) efficient allocation mechanism. As given in the formula for S^Q, the relative efficiency of the waiting-line auction is measured by the function $h(\beta, n, m)$. When $h(\beta, n, m)$ is less (greater) than 1, the waiting-line auction is less (more) efficient than the lottery. There are two factors governing $h(\beta, n, m)$: a scarcity factor (through the ratio m/n) and a distributional shape factor (through β). We consider these factors in turn.

The scarcity factor. In the waiting-line auction, for a given distribution of valuations, the optimal waiting time $\tau(v_i)$ of consumer i is a decreasing function in m and an increasing function in n.[11] When there are more objects available to be allocated (a higher m/n ratio), the intensity of competition is reduced. Each consumer will reduce his optimal waiting time and therefore arrive at the queue later. Similarly, if there are more consumers competing for the objects (i.e. a lower m/n ratio), competition will intensify. Anticipating tougher competition, every consumer will choose to join the queue earlier. Hence, it is no surprise

[11] The derivation of these results are straightforward, and follow standard analysis in the auction literature. They are available on request.

that $h(\beta, n, m)$ have the opposite signs as $\tau(v)$, with respect to changes in m/n. In other words, the relative efficiency of the waiting-line auction, as measured by $h(\beta, n, m)$, improves when the m/n ratio rises, and declines when the ratio falls.

The distributional shape factor. The relative efficiency of the waiting-line auction is dependent on the distribution of valuations. A shift in the distribution affects the optimal waiting time $\tau(v)$, and in turn impacts the rent-seeking costs. When $\beta < 1$, the density function of monetary valuation is L-shaped. From the perspective of a consumer in the waiting-line auction, this means that he is likely to be facing competitors that possess mostly low valuations. With less intense competition, the optimal waiting time will be small. As β increases, the distribution of V shifts to the right in the sense of first-order stochastic dominance. From the perspective of each consumer, this means that there is a greater likelihood that he will be facing competitors that possess higher valuations. Hence, each consumer, anticipating keener competition, will revise his optimal waiting time upwards. This leads to higher rent-seeking costs and is the reason behind the general dominance of the lottery for the case when $\beta \geq 1$. Quantitatively, the function $h(\beta, n, m)$ decreases as β increases, so that the relative efficiency of lottery over waiting-line S^R/SQ increases.

Proposition 1 indicates that when $\beta \geq 1$, the waiting-line auction is unconditionally dominated by the lottery, regardless of the m/n ratio. For the case where $\beta < 1$, when the expected social surplus is smaller and the rent-seeking costs (i.e. waiting times) are lower, the waiting-line auction may possibly dominate the lottery, if the rent-seeking cost is lower than the resource misallocation cost of the lottery. Hence, when $\beta < 1$, the relative scarcity of the objects (as measured by m/n) becomes a critical factor in the relative dominance of the two mechanisms. As stated in Proposition 1, when β falls to $\frac{1}{2}$, the waiting-line auction will dominate the lottery if m/n is greater than $\frac{1}{2}$. Otherwise, the lottery remains the preferred allocation mechanism. In general, if the change in the rent-seeking costs is larger than the change in the expected surplus, the allocative efficiency of the waiting-line auction deteriorates relative to the lottery.

If the rent-seeking cost is sufficiently small, and lower than the resource misallocation cost of the lottery, the waiting-line auction would be the more efficient mechanism.

3.2 Monetary valuation with the Weibull distribution

Next, we consider the Weibull distribution, which is useful, for our purpose, in modeling monetary valuation distributions that are (i) extremely positively-skewed, (ii) unimodal and positively-skewed , and (iii) nearly symmetric. These descriptions correspond, respectively, to the cases where (i) the majority of consumers attach very low valuations to the objects, (ii) the majority of consumers attach low to medium valuations, and (iii) the majority of consumers attach moderate valuations, with a few attaching very low or very high valuations. The Weibull distribution thus extends the range of situations for the comparison of the relative allocative efficiency of the two mechanisms. The cumulative distribution function of the Weibull distribution takes the form

$$F(v;\theta,\beta) = 1 - \exp\left(-\left(\frac{v}{\theta}\right)^{\beta}\right), \ v > 0; \theta > 0, \beta > 0, \tag{7}$$

where β is the shape parameter and θ is the scale parameter. When $\beta \leq 1$, the density function is a decreasing function. When $\beta > 1$, the density function is unimodal with a longer tail to the right.[12] Note that when $\beta = 1$, the Weibull becomes an exponential distribution. The mean and variance of a Weibull random variable are, respectively, $E(V) = \theta\Gamma(1+\frac{1}{\beta})$ and $Var(V) = \theta^2[\Gamma(1+\frac{2}{\beta}) - [\Gamma(1+\frac{1}{\beta})]^2]$. Figure 2 provides illustrations of the density function of the Weibull distribution.

Figure 2

Lemma 2: *If consumers' monetary valuations are drawn from the Weibull distribution, the expected social surplus functions associated with the two allocation mechanisms are given by $S^R = m\theta\Gamma(1+1/\beta)$ and $S^Q = S^R h(\beta,n,m)$, respectively, where,*

$$h(\beta,n,m) = \frac{n}{m}\sum_{n-m}^{n-1}\sum_{j=0}^{k}\binom{n-1}{k}\binom{k}{j}(-1)^j\left(\frac{1}{n-k+j}\right)^{1/\beta}. \tag{8}$$

Furthermore, $h(\beta,n,m)$ is decreasing in β and $h(1,n,m) = 1$.

Proposition 2: *Suppose consumers' monetary valuations are drawn from the Weibull distribution. The lottery dominates the waiting-line auction if $\beta > 1$; the waiting-line dominates the lottery if $\beta < 1$; the two allocation mechanisms are equally efficient when $\beta = 1$, which represents the case where monetary valuations follow an exponential distribution.*

The results of Lemma 2 and Proposition 2 are rather striking. They show that when monetary valuations can be modeled as a Weibull distribution, the dominance of the lottery over the waiting-line auction, and vice versa, depends only on the shape parameter β of the distribution. More significantly, it does not depend on the scarcity factor, as measured by the m/n ratio.

[12] It can be shown that when $\beta = 3.768$, the density function of the Weibull distribution is very similar to that of a normal distribution (See Hernandez and Johnson (1980)).

While the dominance of one allocation mechanism over the other is completely characterized by the value of β, the *degree* of dominance still depends on the scarcity factor m/n. To see this, we plot the relative efficiency of the lottery over the waiting-line auction (i.e., $S^R/S^Q = 1/h(\beta, n, m)$) in Figure 3. From the plot of S^R/S^Q against β (left-hand plot of Figure 3) for given m and n, we note that as β increases, the relative efficiency of the lottery over the waiting-line auction improves most rapidly when the ratio m/n is smallest. As the ratio m/n increases, the curve of S^R/S^Q flattens, so that when $m/n \approx 1$, the ratio $S^R/S^Q \approx 1$. Next, from the plot of S^R/S^Q vs m for given β and n (right-hand plot of Figure 3), we see that the ratio S^R/S^Q approaches 1 when m approaches n. These results suggests that when the scarcity of the objects to be allocated is not an issue (i.e. when the ratio $m/n \approx 1$), it does not matter too much which allocation mechanism is used, as the difference in allocative efficiency will likely be small. Conversely, when the ratio m/n declines, there will be a marked discrepancy in the allocative efficiency of the two mechanisms.

The intuition behind these comparative results lie in the fact that as m/n falls, the scarcity factor forces consumers to raise their optimal waiting time, thereby increasing the rent-seeking costs of the waiting-line auction. By contrast, the lottery does not suffer from rent dissipation through rent-seeking costs; hence, its dominance over the waiting-line auction improves when the ratio m/n declines.

Figure 3

3.3 Monetary valuation with the logistic distribution

We include the class of logistic distributions in our analysis for the following reason: in a variety of economic settings, it is reasonable to approximate the distribution of monetary valuations as a symmetric distribution. The logistic distribution function serves this purpose, and at the same time enables us to derive closed-form expressions for the expected surplus functions.[13] The cumulative distribution function of the logistic distribution has the following form

$$F(v; \mu, \theta) = 1 - \frac{1}{1 + \exp[(v - \mu)/\theta]}, \ -\infty < v < \infty; -\infty < \mu < \infty, \theta > 0. \tag{9}$$

Note that the logistic distribution is symmetric around the mean $E(V) = \mu$ with variance $Var(V) = \frac{1}{3}\pi^2\theta^2$. Note that μ is a location parameter and θ is the scale parameter. The larger θ is, the flatter is the pdf. A few plots of the logistic pdf are provided in Figure 4 for illustrative purpose.

[13]We chose the class of logistic distributions over the class normal distributions for our analysis, as the latter class of distributions does not allow us to derive closed-form expressions for the expected surplus functions. With suitable choice of parameters, the logistic distribution may approximate a normal distribution.

Figure 4

Lemma 3: *If consumers' monetary valuations are drawn from the logistic distribution, the expected social surplus functions associated with the two allocation mechanisms are*

$$S^R = m\mu \text{ and } S^Q = n\theta[\Psi(n) - \Psi(n-m)]$$

where $\Psi(\cdot)$ *is the digamma function defined as* $\Psi(z) = d\log\Gamma(z)/dz$*.*

Note that the logistic distribution may assume negative values, which has no economic meaning for the problem at hand, since an individual with a negative monetary valuation will not choose to participate in either the lottery or the waiting-line auction. However, we can make the probability of negative values negligible by having a large mean-to-scale ratio, i.e., $\mu/\theta \geq 10$.[14] Taking $\mu/\theta = 10$, we have

$$\frac{S^Q}{S^R} = \frac{n}{10m}[\Psi(n) - \Psi(n-m)],$$

which is an increasing function of m. It can be easily verified that $\max_m(S^Q/S^R) < 1$ for $n < 10000$. Hence, we can conclude that if monetary valuations follow a logistic distribution, the lottery mechanism essentially dominates the waiting-line auction.

Proposition 3: *If consumers' monetary valuations are drawn from the logistic distribution with a negligible probability of negative valuations (*$\mu/\theta \geq 10$*), the lottery almost always dominates the waiting-line auction.*

This result is particularly striking as it suggests that when the monetary valuations is modeled as a symmetric distribution and time costs are homogeneous, the optimal allocation mechanism is almost always a random allocation, regardless of the number of objects to be allocated or the number of participants.

3.4 Monetary valuation with the beta distribution

The final class of distributions that we shall consider is the beta distribution. The probability density function (pdf) of the beta distribution has the following form

$$F(v;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}v^{\alpha-1}(1-v)^{\beta-1},\ 0 \leq v \leq 1, \alpha > 0, \beta > 0. \tag{10}$$

In terms of the potential shapes of the density function, the beta distribution is the richest family of distributions. It is U-shaped if $\alpha < 1$ and $\beta < 1$, uniform if $\alpha = 1$ and $\beta = 1$,

[14] The probability of negative values is $F(0,\mu,\theta) = 1 - 1/[1 + \exp(-\mu/\theta)]$. When $\mu/\theta \geq 10$, we have $F(0,\mu\theta) \leq 4.5 \times 10^{-5}$.

L-shaped if $\alpha < 1$ and $\beta > 1$, J-shaped if $\alpha > 1$ and $\beta < 1$, and unimodal, otherwise. Furthermore, when $\beta = 1$, the beta distribution becomes a special case of the power function distribution. The mean and variance of the beta distribution are $E(V) = \frac{\alpha}{\alpha+\beta}$ and $Var(V) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, respectively. We provide plots of the beta pdf in Figure 5.

Figure 5

As the cumulative distribution function of the beta distribution does not have a closed-form expression, it is difficult to derive general closed-form functions for the expected surplus functions of the lottery and the waiting line auction. For the purpose of our study, we have elected to conduct our analysis in the following manner. First, we consider two special cases of the beta distribution - when $\alpha = 1$ and when $\beta = 1$ - which allow us to obtain closed-form expression of the ratio S^R/S^Q. The analysis of these two cases allow us to obtain a set of sharp comparative results. Next, for the general case where no analytical solutions are available, we compute the values of the ratio S^R/S^Q (using the general expression given in (4)) for a range of parameter configurations of β, n and m. These results are presented in Figure 6.

To begin with, note that for the special case when $\beta = 1$, the distribution becomes a power function distribution; hence, the results of Lemma 1 apply. In particular, we conclude that when $\beta = 1$, the lottery dominates the waiting-line auction if $\alpha \geq 1$, and if $\alpha < 1$ it continues to dominate the waiting-line auction provided that m/n is sufficiently small. Similarly, for $\beta > 1$, the lottery remains the more efficient allocation mechanism. For the second special case when $\alpha = 1$, we first prove the following lemma.

Lemma 4: *If consumers' monetary valuations are drawn from the beta distribution and* $\alpha = 1$*, the expected social surplus functions associated with the two allocation mechanisms are* $S^R = m/(1+\beta)$ *and* $S^Q = S^R h(\beta, n, m)$*, respectively, where*

$$h(\beta, n, m) = \frac{n!\Gamma(m+1+1/\beta)}{m!\Gamma(n+1+1/\beta)}. \tag{11}$$

Furthermore, $h(\beta, n, m)$ *is increasing in* β*, and* $\lim_{\beta\to\infty} h(\beta, n, m) = 1$.

The implication of Lemma 4 is that when $\alpha = 1$,the lottery dominates the waiting-line auction regardless of the value of β, or the ratio m/n. This conclusion can be generalized straightforwardly to the case when $\alpha > 1$. The intuition here is that, as α increases from 1, the weight of the pdf shifts to the right, so the distribution shifts in terms of first-order stochastic dominance. In turn, this means a greater likelihood that consumers participating in the waiting-line auction possess higher valuations. Thus, competition intensifies in the waiting-line auction and each consumer will raise her optimal waiting time. Therefore, rent-seeking costs are higher when the distribution shifts to the right. As a result, the $h(\beta, n, m)$

function decreases as α increases (see Figure 6). Combining Lemmas 1 and 4, and the above discussions, we can conclude that

Proposition 4: *If consumers' monetary valuations are drawn from the beta distribution, the lottery dominates the waiting-line auction, except possibly when* $\alpha < 1$, $\beta \geq 1$, *i.e., when the beta distribution is L-shaped, and* m/n *is sufficiently large.*

Figure 6

3.5 Generalization of results for homogeneous consumers

The analysis in the preceding sub-sections has identified a set of conditions under which the lottery dominates the waiting-line auction. Let us summarize the results here. For the power function, the Weibull, the logistic and the beta functions, we note that as long as the weight of the pdf is not extremely concentrated on the left-hand side, the lottery is the more efficient mechanism. In fact, when the pdf is unimodal, U-shaped, J-shaped or uniform, the lottery is the more efficient allocation scheme.

It is only in the case when the pdf is L-shaped that the waiting-line auction may dominate the lottery. In the case of the Weibull distribution, this occurs when the shape parameter β is less than 1, independent of the scarcity factor m/n. In the case of the power function distribution and the beta distribution, the dominance of the waiting-line auction depends also on a sufficiently high m/n ratio. When the m/n ratio is very small, the lottery continues to dominate, as we showed in an example for the case of the power function distribution. Thus, while the waiting-line auction may dominate the lottery in some situations, the conditions under which this can occur are very stringent. These conditions are: a sufficiently low likelihood of high-valuation individuals participating in the waiting-line auction, and a sufficiently large numbers of objects for allocation (relative to the number of participants) so that the scarcity factor is not an issue.

The intuition here is that when consumers are sufficiently homogeneous and possess low valuations, and the ratio m/n is sufficiently high, the lower degree of resource misallocation in the waiting-line auction (relative to the lottery) more than compensates for the rent-seeking costs of waiting in the queue. We summarize these findings into the following Proposition.

Proposition 5: *The lottery dominates the waiting-line auction for all time valuation distributions that are unimodal, U-shaped, J-shaped, uniform, etc., independent of the scarcity factor (as measured by* m/n*). The waiting-line auction may dominate the lottery when the density distributions are L-shaped, provided that the ratio* m/n *is sufficiently high.*

4 Efficiency Comparison: Heterogeneous Consumers

In the previous section, we analyzed the case where consumers' time costs are relatively homogeneous, so that the assumption of identical time costs is reasonable. When consumers' time costs vary significantly, it is more appropriate to consider a joint distribution for time valuations Y and time costs W or a joint distribution for monetary valuation V and time cost W. When time costs and time valuations are correlated, rent dissipation in the waiting-line auction includes potential resource misallocation as well. We first note a couple of interesting cases : (i) V and W are independent and (ii) Y and W are independent. Using the general expression given in (3), some simple conditioning arguments show that when W is independent of V, all the results of Lemmas 1 to 4 go through. Again, using (3), a direct manipulation shows that the S^R/S^Q remains the same if W is no longer constant but is independent of Y. This allows us to draw a general conclusion that the heterogeneity in time costs does not change the relative efficiency of the lottery over the waiting-line auction if time costs are independent of monetary valuations or time valuations.

For the more general case, we present a model of joint distribution between time valuations and time costs to analyze the impact of heterogeneity on the relative allocative efficiency.[15] Our analysis suggests that the relative efficiency of the waiting-line auction in fact improves when there is positive correlation in time costs and monetary valuations, and deteriorates when the correlation is negative. When the correlation is zero, the relative efficiency of the waiting-line auction is not affected by heterogeneity in time costs.

4.1 Positively correlated time valuation and time cost

Consider the case where Y and W are uniformly distributed on the area $A = \{(y, w) : [0 \le y \le 1], [0 \le w \le \beta y^{\beta-1}]\}$. The joint pdf of Y and W is

$$f(y, w) = 1, \qquad 0 \le y \le 1,\ 0 \le w \le \beta y^{\beta-1};\ \beta \ge 1. \tag{12}$$

It is easy to verify that $f_Y(y) = \beta y^{\beta-1}, 0 \le y \le 1$, i.e., the marginal distribution of Y is the power function distribution with the scale parameter equal to 1, and $f_W(w) = 1 - (w/\beta)^{\frac{1}{\beta-1}}, 0 \le w \le \beta$. The correlation coefficient between Y and W is

$$\rho(Y, W) = \frac{\beta - 1}{2\beta}\sqrt{\frac{(\beta+2)(9\beta-6)}{7\beta^2 - 2\beta + 4}}$$

For $\beta = 1, 2, 4, \infty$, we have $\rho(Y, W) = 0, 0.32, 0.48$, and 0.57, respectively. This shows that Y and W are uncorrelated when $\beta = 1$, and that the correlation increases as β increases (with an upper limit of 0.57).

[15] From a modeling point of view, we could consider either a joint distribution of time valuations and time costs or a joint distribution of monetary valuations and time costs. As our motivation here is to demonstrate the impact of heterogeneity on relative efficiency, we have chosen the current specification for its tractability.

Lemma 5: *When time valuation Y and time cost W are drawn jointly from the distribution specified in (12), the expected social surplus functions are $S^R = \frac{1}{4}m\beta$, and $S^Q = S^R h(\beta, n, m)$, where,*

$$h(\beta, n, m) = \frac{2\beta}{2\beta - 1}\left(\frac{n}{m} - \frac{1}{\beta} + \frac{m+1}{2\beta(n+1)} - \frac{n!\Gamma(n-m+\frac{1}{\beta})}{m(n-m-1)!\Gamma(n+\frac{1}{\beta})}\right) \tag{13}$$

Furthermore, $h(\beta, n, m)$ is decreasing in β, $h(1, n, m) = \frac{m+1}{n+1}$ and $\lim_{\beta\to\infty} h(\beta, n, m) = 0$.

A direct comparison of the h functions, defined in Lemma 1 and Lemma 5 allows us to determine the effect of positive correlation of time costs and time valuations on relative allocative efficiency. First, note that when $\beta = 1$, both h functions have the same value. This implies that when time valuations and time costs are uncorrelated, the relative allocative efficiency of the waiting-line auction versus the lottery remains unchanged. Therefore, the results of Proposition 1 continues to hold even if consumers are heterogeneous in their time costs. When β tends to ∞, both the h functions are trivially identical, as both h functions tend to zero.

More importantly, as β increases – causing the marginal distribution of time valuations $f_Y(y)$ to become more negatively skewed – the waiting-line auction remains dominated by the lottery, regardless of the degree of correlation between Y and W. This result follows directly from Lemma 5 that $h(\beta, n, m) \leq 1$ when $\beta = 1$ and is decreasing in β. Furthermore, it is easy to verify that the degree of dominance of the lottery over the waiting-line auction increases with β, and decreases with m/n. The set of results indicate that the general dominance of the lottery extends to the case where time costs are positively correlated with time valuations.

4.2 Negatively correlated time valuation and time cost

To analyze the impact of negative correlation on relative allocative efficiency, consider the following specification of time cost: $W^* = \beta - W$.[16] Then,

$$\rho(Y, W*) = -\rho(Y, W) = -\frac{\beta - 1}{2\beta}\sqrt{\frac{(\beta+2)(9\beta-6)}{7\beta^2 - 2\beta + 4}},$$

i.e., the time valuation Y and time cost W^* are negatively correlated.

Lemma 6: *If Y and W follow the joint distribution specified in (12), the expected social surplus functions under the time valuation Y and time cost $W^* = \beta - W$ are $S^R =$*

[16] This specification is chosen for its tractability and does not affect the generality of the results presented here.

$\frac{1}{4}m\beta$*, and* $S^Q = S^R h(\beta, n, m)$*, where,*

$$h(\beta, n, m) = \frac{4\beta}{3\beta - 1} h_1(\beta, n, m) - \frac{\beta + 1}{3\beta - 1} h_2(\beta, n, m) \tag{14}$$

with h_1 *being the* h *function defined in Lemma 1 and* h_2 *the* h *function defined in Lemma 5. Furthermore,* $h(\beta, n, m)$ *is decreasing in* β*,* $h(1, n, m) = \frac{m+1}{n+1}$ *and* $\lim_{\beta \to \infty} h(\beta, n, m) = 0$*.*

Together, Lemmas 5 and 6 allow us to compare the relative efficiency of the waiting-line auction under the alternative scenarios of positive and negative correlations of time costs and time valuations. We shall conduct the comparison by way of a graphical analysis, as provided in Figure 7.

Figure 7

In Figure 7, we have plotted the three h functions (defined in Lemmas 1, 5 and 6), against m, for a given value of β and for $n = 50$. The plots in Figure 7 show that when Y and W are positively correlated, the relative efficiency of the waiting-line auction over the lottery is higher than in the case when Y and W are un-correlated. The converse is true when Y and W are negatively correlated.

It is important to note that even though the relative efficiency of the waiting-line auction over the lottery improves in the case of positive correlation between time costs and time valuations, the lottery remains the more efficient mechanism (as noted in the discussion following Lemma 5) in the model that we present here. We summarize our findings in the following Proposition.

Proposition 6: *If consumers' time valuations* Y *and time costs* W *follow the joint distribution specified in (12), the allocative efficiency of the waiting-line auction improves (deteriorates) if* Y *and* W *are more positively (negatively) correlated, compared with the case when they are uncorrelated.*

Note that since monetary valuation is a product of time valuation and time cost, $V = YW$, a positive correlation between Y and W implies a positive correlation between V and W. Similarly, it is straightforward to show that V and W^* are negatively correlated.

A common perception (and argument against the use of the waiting-line auction as an allocation mechanism) is that less wealthy or financially constrained individuals with lower ability to pay (i.e. monetary valuations) are also likely to have lower time-costs. The assertion is that these individuals are likely to join the queue earlier in a waiting-line auction, so that the objects are not necessarily allocated to those who might possess higher monetary valuations. The equivalent assertion is that when time valuations Y and time costs W are positively correlated, the allocative efficiency of the waiting-line auction is lower than when time costs are uncorrelated (or negatively correlated) with time valuations. Proposition 6 suggests that, contrary to conventional wisdom, the efficiency of the waiting-line auction in our model in fact improves when opportunity cost of time is positively correlated with monetary valuations.

It would be of interest to consider some more general joint distributions for time valuations and time costs. However, in light of the strong results derived in this section, and combined with the general analysis given in the earlier sections, we believe that although heterogeneity in time costs may result in an improvement in the relative efficiency of the waiting-line auction over the lottery, the improvement is not likely to be significant enough to reverse the general dominance of the lottery over the waiting-line auction, unless the marginal distribution of time valuation is extremely positively skewed (i.e. L-shaped).

5 Conclusion

This paper set out to analyze the relative allocative efficiency of the waiting-line auction and the lottery. By comparing the expected social surplus functions of the two mechanisms, we are able to delineate the circumstances under which a random allocation mechanism is more efficient than the waiting-line auction, and vice versa. Our analysis suggests that when time costs are homogeneous, the lottery is the optimal mechanism in a wide range of circumstances (Propositions 1 to 5). We also analyzed the case when time costs are heterogeneous and correlated with time valuations. We are able to show that the relative efficiency of the waiting-line auction improves when there is positive correlation between time costs and time valuations, but deteriorates when the correlation is negative (Proposition 6). Our results indicate that besides its *equity* appeal, the lottery mechanism is also the *efficient* non-price allocation mechanism in a wide variety of situations.

While we assumed in our analysis that consumers are risk-neutral and no secondary market exists, our results continue to hold when these assumptions are relaxed. If there are no restrictions on transferability, the resource misallocation in both the lottery and the waiting-line auction can potentially be eliminated, as successful individuals with low valuations transfer, for a fee, their allocation to those with higher valuations. However, rent-seeking cost in the waiting-line auction will increase if there are resale opportunities, as individuals, anticipating keener competition, will choose to arrive at the queue earlier. It is unclear if the reduction in resource misallocation would more than compensate for the increase in rent-seeking costs when a secondary market exists.

Suppose consumers are risk averse instead of risk-neutral. As is well-known in the auction literature (see Maskin and Riley (1984)), it can be shown that the optimal waiting time, and consequently the rent-seeking costs, will also be higher. Furthermore, if there is uncertainty over the number of objects to be allocated and/or and the number of potential participants (i.e. the scarcity factor m/n is not common knowledge), introducing risk aversion will also lead to a longer optimal waiting time.

While the analysis in this paper is *positive* and does not address the issue of equity, we hope the results presented here will contribute to a better understanding among policy-makers on the choice of the appropriate non-price allocation mechanism. Clearly, if more weight is assigned to the welfare (expected payoffs) of a particular group of individuals, the relative desirability of two allocation schemes may not follow the ranking based on allocative efficiency. Specifically, it is conceivable that if it is desirable that the allocation favors, say, the lower-income group, and time costs and positively correlated with time valuation, then a waiting-line auction may be the preferred allocation scheme. However, a potentially better allocation scheme may be to segregate the n participants into two or more groups, based on income, and conduct separate lotteries for the different groups. Relatively more of the m objects could be alloted to the lower-income group. This meets the objective of favoring the lower-income group, without incurring rent-seeking costs of waiting in line.

We conclude this paper with a conjecture regarding the applicability of our findings to contests where rent-seeking costs are significant in determining the outcomes. Specifically, consider R&D races in which patent rights are granted on the basis of (nearly) completed invention; this type of R&D race is akin to waiting-line auction for patent allocation. The competing firms have different monetary valuations that they attach to the patent rights and their cost of entering the R&D race (akin to time costs) may be correlated with their valuations. If there are only a few patents to be awarded and there are a large number of competing firms, the level of rent-seeking costs is likely to be significant.

Our results in this paper suggests that introducing an element randomization in the allocation process may perhaps alleviate rent-seeking costs and improve the efficiency of the R&D race. In an R&D race, technical progress is usually stochastic and the leaders at the early stage of the invention process may not be the eventual winners. Thus, if patent rights are awarded at an earlier stage of the invention process, say, to interim leaders, this may serve to reduce the overall rent-seeking costs. In this sense, the patent allocation process can incorporate a random element. A completely random allocation of patent rights is, of course, undesirable in most circumstances since the granting of one set of patent rights may affect the innovative processes in the future and, consequently, the pace of economic growth. An important consideration is that randomization in the allocation of patent rights may reduce the incentives to innovate and affect the rate of invention in equilibrium. Thus, there is a tradeoff between the potential reduction in rent-dissipation, when a random element is incorporated in the allocation process, and the dis-incentives that may be created as a result. This is an important issue for further research.

Appendix: Proofs of the Lemmas

The proofs of Lemmas 1-4 are basically the derivations of S^Q give in (4) in terms of money valuation V. That is,

$$S^Q = n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \int_{\underline{v}}^{\overline{v}} F_V(v)^k [1 - F_V(v)]^{n-k} dv.$$

Proof of Lemma 1. With the power function distribution, we have $F_V(v) = (v/\theta)^\beta$, and

$$\begin{aligned}
S^R &= mE(V) = \frac{m\theta\beta}{1+\beta} \\
S^Q &= n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \int_0^\theta [(v/\theta)^\beta)]^k [1 - (v/\theta)^\beta]^{n-k} dv \\
&= n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \frac{\theta}{\beta} \int_0^1 u^{k+\frac{1}{\beta}-1} (1-u)^{n-k} du, \quad \text{(by letting } u = (v/\theta)^\beta) \\
&= n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \frac{\theta}{\beta} \frac{\Gamma(k+\frac{1}{\beta})\Gamma(n-k+1)}{\Gamma(n+1+\frac{1}{\beta})} \\
&= \frac{\theta}{\beta} \frac{\Gamma(n+1)}{\Gamma(n+1+\frac{1}{\beta})} \sum_{k=n-m}^{n-1} \frac{\Gamma(k+\frac{1}{\beta})}{k!}(n-k) \\
&= \frac{\theta}{\beta} \frac{\Gamma(n+1)}{\Gamma(n+1+\frac{1}{\beta})} \left(\frac{\beta^2 \Gamma(n+1+\frac{1}{\beta})}{(1+\beta)\Gamma(n)} - \frac{\beta(\beta n + m + 1)\Gamma(n-m+\frac{1}{\beta})}{(1+\beta)\Gamma(n-m)} \right) \\
&= \frac{m\theta\beta}{1+\beta} \left(\frac{n}{m} - \frac{n!(\beta n + m + 1)\Gamma(n-m+\frac{1}{\beta})}{\beta m (n-m-1)! \Gamma(n+1+\frac{1}{\beta})} \right).
\end{aligned}$$

With the assistance of Mathematica, it is straightforward to verify the properties of the h function stated in the Lemma.

Proof of Lemma 2. With the Weibull distribution, we have $F_V(v) = 1 - \exp[-(v/\theta)^\beta]$, and

$$\begin{aligned}
S^R &= mE(V) = m\theta\Gamma(1 + \frac{1}{\beta}) \\
S^Q &= n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \int_0^\infty [1 - \exp(-(v/\theta)^\beta)]^k [\exp(-v/\theta)^\beta]^{n-k} dv.
\end{aligned}$$

Making a change of variable $u = (v/\theta)^\beta$, and then applying the binominal expansion to $[1 - \exp(-u)]^k$, the integral in the summation for S^Q becomes

$$\int_0^\infty [1 - \exp(-(v/\theta)^\beta)]^k [\exp(-v/\theta)^\beta]^{n-k} dv$$

$$
\begin{aligned}
&= \frac{\theta}{\beta}\int_0^\infty u^{1/\beta-1}[1-\exp(-u)]^k[\exp(-u)]^{n-k}du \\
&= \frac{\theta}{\beta}\int_0^\infty u^{1/\beta-1}\left\{\sum_{j=0}^{k}\left(\begin{array}{c} k \\ j \end{array}\right)(-1)^j\exp(-ju)\right\}[\exp(-(n-k)u)]du \\
&= \frac{\theta}{\beta}\int_0^\infty u^{1/\beta-1}\left\{\sum_{j=0}^{k}\left(\begin{array}{c} k \\ j \end{array}\right)(-1)^j\exp(-(n-k+j)u)\right\}du \\
&= \frac{\theta}{\beta}\sum_{j=0}^{k}\left(\begin{array}{c} k \\ j \end{array}\right)(-1)^j\int_0^\infty u^{1/\beta-1}\exp[-(n-k+j)u]du \\
&= \frac{\theta}{\beta}\sum_{j=0}^{k}\left(\begin{array}{c} k \\ j \end{array}\right)(-1)^j\Gamma(1/\beta)\left(\frac{1}{n-k+j}\right)^{1/\beta}.
\end{aligned}
$$

Substituting this back into the expression for S^Q, we have

$$
\begin{aligned}
S^Q &= n\frac{\theta}{\beta}\sum_{k=n-m}^{n-1}\left(\begin{array}{c} n-1 \\ k \end{array}\right)\sum_{j=0}^{k}\left(\begin{array}{c} k \\ j \end{array}\right)(-1)^j\Gamma(1/\beta)\left(\frac{1}{n-k+j}\right)^{1/\beta} \\
&= n\theta\Gamma(1+1/\beta)\sum_{k=n-m}^{n-1}\sum_{j=0}^{k}\left(\begin{array}{c} n-1 \\ k \end{array}\right)\left(\begin{array}{c} k \\ j \end{array}\right)(-1)^j\left(\frac{1}{n-k+j}\right)^{1/\beta} \\
&= m\theta\Gamma(1+1/\beta)h(\beta,n,m).
\end{aligned}
$$

Since $1/(n-k+j)\le 1$ with equality occurring only when $k=n-1$, and $j=0$, the terms in the summation of $h(\beta,n,m)$ are thus either constant or increasing as β increases. Hence h is an increasing function of β. Finally,

$$
\begin{aligned}
h(1,n,m) &= \frac{n}{m}\sum_{k=n-m}^{n-1}\left(\begin{array}{c} n-1 \\ k \end{array}\right)\sum_{j=0}^{k}\left(\begin{array}{c} k \\ j \end{array}\right)(-1)^j\left(\frac{1}{n-k+j}\right) \\
&= \frac{n}{m}\sum_{k=n-m}^{n-1}\left(\begin{array}{c} n-1 \\ k \end{array}\right)\frac{k!}{(n-k)(n-k+1)\cdots(n-1)n} \\
&= \frac{n}{m}\sum_{k=n-m}^{n-1}\frac{1}{n}=1.
\end{aligned}
$$

Note that the first summation is handled by a combinatory formula

$$
\sum_{j=0}^{k}\left(\begin{array}{c} k \\ j \end{array}\right)\frac{(-1)^j}{a+j}=\frac{k!}{a(a+1)\cdots(a+k)},\quad \text{for } a\neq 0,-1,-2,\cdots,-k.
$$

Proof of Lemma 3. With the logistic distribution, we have

$$
S^R = mE(V)=m\mu
$$

$$
\begin{aligned}
& \int_{-\infty}^{\infty} F_V(v)^k [1 - F_V(v)]^{n-k} dv \\
= & \int_{-\infty}^{\infty} \left[1 - \frac{1}{1+\exp((v-\mu)/\theta)}\right]^k \left[\frac{1}{1+\exp((v-\mu)/\theta)}\right]^{n-k} dv.
\end{aligned}
$$

Letting $w = \{1 + \exp[(v-\mu)/\theta]\}^{-1}$, the above integral becomes

$$
\begin{aligned}
& \int_0^1 (1-w)^k w^{n-k} \frac{\theta}{w(1-w)} dw \\
= & \ \theta \int_0^1 (1-w)^{k-1} w^{n-k-1} dw \\
= & \ \theta \frac{\Gamma(k)\Gamma(n-k)}{\Gamma(n)}
\end{aligned}
$$

This gives

$$
\begin{aligned}
S^Q &= n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \frac{\theta\Gamma(k)\Gamma(n-k)}{\Gamma(n)} \\
&= n\theta \sum_{k=n-m}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} \frac{(k-1)!(n-k-1)!}{(n-)!} \\
&= n\theta \sum_{k=n-m}^{n-1} \frac{1}{k} = n\theta[\Psi(n) - \Psi(n-m)].
\end{aligned}
$$

Proof of Lemma 4. When $\alpha = 1$, the beta distribution becomes $F_V(v) = 1 - (1-v)^\beta$. This gives

$$
\begin{aligned}
S^R &= mE(V) = \frac{m}{1+\beta} \\
S^Q &= n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \int_0^1 [1-(1-v)^\beta]^k (1-v)^{\beta(n-k)} dv \\
&= n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \frac{1}{\beta} \frac{\Gamma(n-k+\frac{1}{\beta})}{\Gamma(n+1+\frac{1}{\beta})} \\
&= \frac{n!}{\beta\Gamma(n+1+\frac{1}{\beta})} \sum_{k=n-m}^{n-1} \frac{\Gamma(n-k+\frac{1}{\beta})}{\Gamma(n-k)} \\
&= \frac{n!}{\beta\Gamma(n+1+\frac{1}{\beta})} \frac{\beta\Gamma(m+1+\frac{1}{\beta})}{(1+\beta)\Gamma(m)}, \quad \text{(by Mahthematica)} \\
&= \frac{m}{1+\beta} \frac{n!\Gamma(m+1+\frac{1}{\beta})}{m!\Gamma(n+1+\frac{1}{\beta})}
\end{aligned}
$$

The rest of the proof is straightforward.

Proof of Lemma 5.

$$\begin{aligned}
S^Q &= nE\left(W\int_0^Y H_Y^Q(x)dx\right)\\
&= n\int_0^1\int_0^{\beta y^{\beta-1}} w\left(\int_0^y H_Y^Q(x)dx\right)f(y,w)dwdy\\
&= n\int_0^1\left(\int_0^y H_Y^Q(x)dx\right)\frac{1}{2}\beta^2 y^{2(\beta-1)}dy\\
&= \frac{n\beta^2}{2}\int_0^1\left(\int_x^1 y^{2(\beta-1)}dy\right)H_Y^Q(x)dx\\
&= \frac{n\beta^2}{2(2\beta-1)}\int_0^1(1-x^{2\beta-1})H_Y^Q(x)dx\\
&= \frac{n\beta^2}{2(2\beta-1)}\sum_{k=n-m}^{n-1}\left(\begin{array}{c} n-1\\ k\end{array}\right)\int_0^1(1-x^{2\beta-1})(x^\beta)^k(1-x^\beta)^{n-k-1}dx, \quad \text{(letting } u=x^\beta)\\
&= \frac{n\beta}{2(2\beta-1)}\sum_{k=n-m}^{n-1}\left(\begin{array}{c} n-1\\ k\end{array}\right)\left(\int_0^1 u^{k+\frac{1}{\beta}-1(1-u)^{n-k-1}}-\int_0^1 u^{k+1}(1-u)^{n-k-1}\right)du\\
&= \frac{n\beta}{2(2\beta-1)}\sum_{k=n-m}^{n-1}\left(\begin{array}{c} n-1\\ k\end{array}\right)\left(\frac{\Gamma(k+\frac{1}{\beta})\Gamma(n-k)}{\Gamma(n+\frac{1}{\beta})}-\frac{\Gamma(k+2)\Gamma(n-k)}{\Gamma(n+2)}\right)\\
&= \frac{\beta}{2(2\beta-1)}\sum_{k=n-m}^{n-1}\left(\frac{\Gamma(n+1)\Gamma(k+\frac{1}{\beta})}{\Gamma(n+\frac{1}{\beta})\Gamma(k+1)}-\frac{k+1}{n+1}\right)\\
&= \frac{\beta}{2(2\beta-1)}\left(\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{\beta})}\left(\frac{\beta\Gamma(n+\frac{1}{\beta})}{\Gamma(n)}-\frac{\beta\Gamma(n-m+\frac{1}{\beta})}{\Gamma(n-m)}\right)-\frac{m(n+1)-\frac{1}{2}m(m+1)}{n+1}\right)\\
&= \left(\frac{m\beta}{4}\right)\frac{2\beta}{2\beta-1}\left(\frac{n}{m}-\frac{1}{\beta}+\frac{m+1}{2\beta(n+1)}-\frac{n!\Gamma(n-m+\frac{1}{\beta})}{m(n-m-1)!\Gamma(n+\frac{1}{\beta})}\right) \quad \#
\end{aligned}$$

With the assistance of Mathematica, it is straightforward to verify the properties of the h function stated in the Lemma.

Proof of Lemma 6.

$$\begin{aligned}
S^Q &= nE\left(W^*\int_0^Y H_Y^Q(x)dx\right) = nE\left((\beta-W)\int_0^Y H_Y^Q(x)dx\right)\\
&= n\beta E\left(\int_0^Y H_Y^Q(x)dx\right) - nE\left(W\int_0^Y H_Y^Q(x)dx\right)
\end{aligned}$$

The first part can be obtained from Lemma 1 and the second part can be obtained from Lemma 5.

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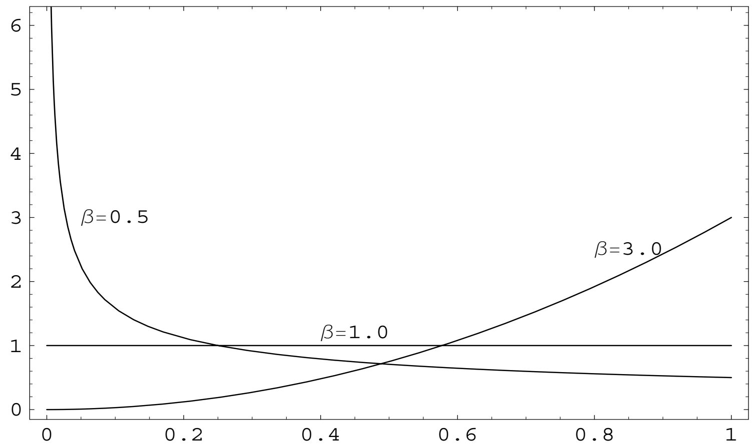


Figure 1: Plots of pdf of power function distribution: $\theta = 1.0$

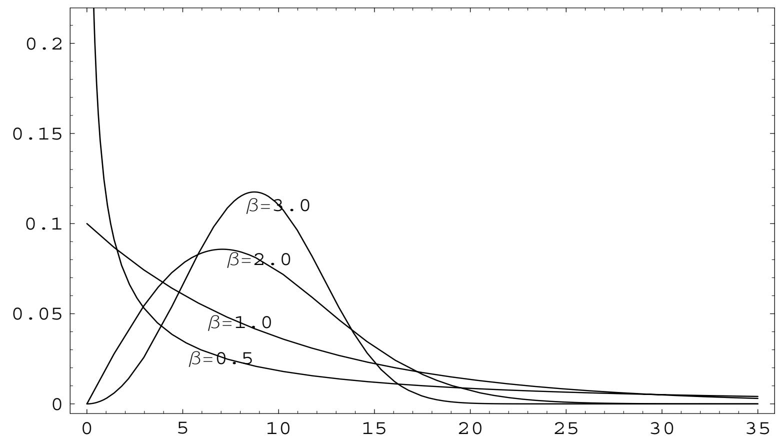


Figure 2: Plots of pdf of Weibull distribution: $\theta = 10.0$

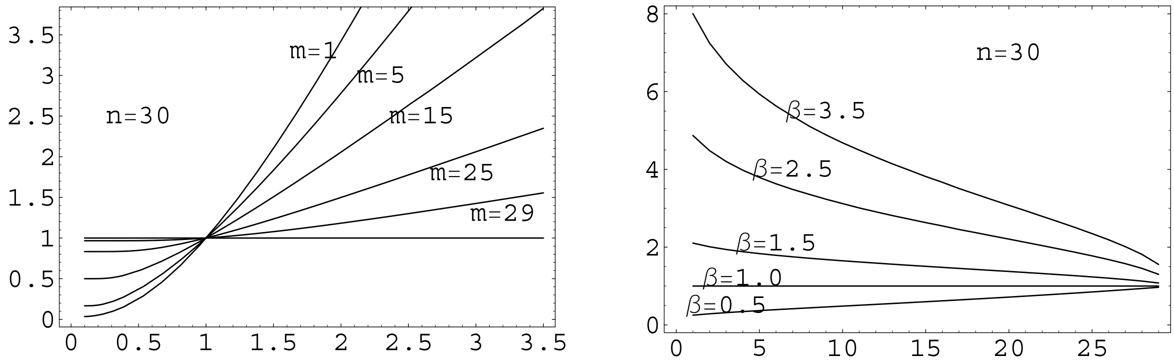


Figure 3: Plots of S^R/S^Q vs β (left), and vs m for Weibull distribution

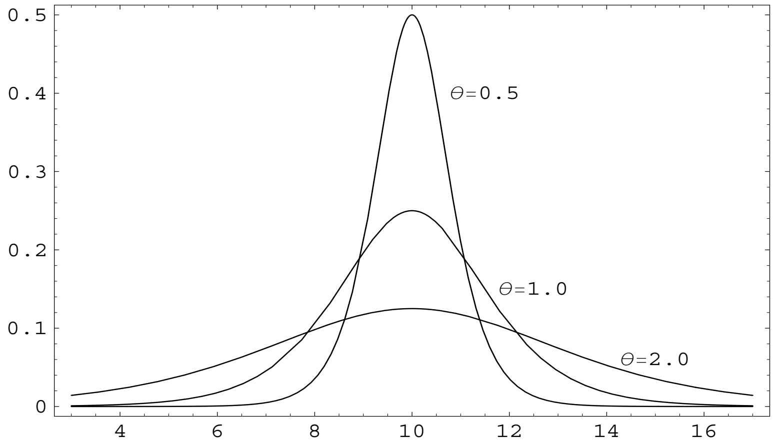
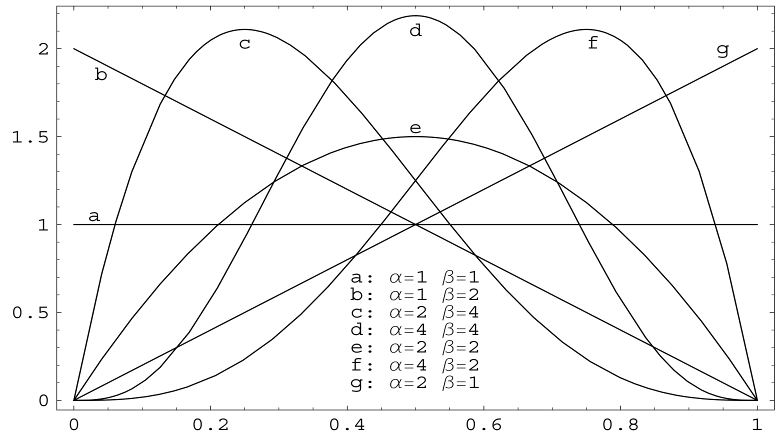


Figure 4: Plots of pdf of logistic distribution: $\mu = 10.0$



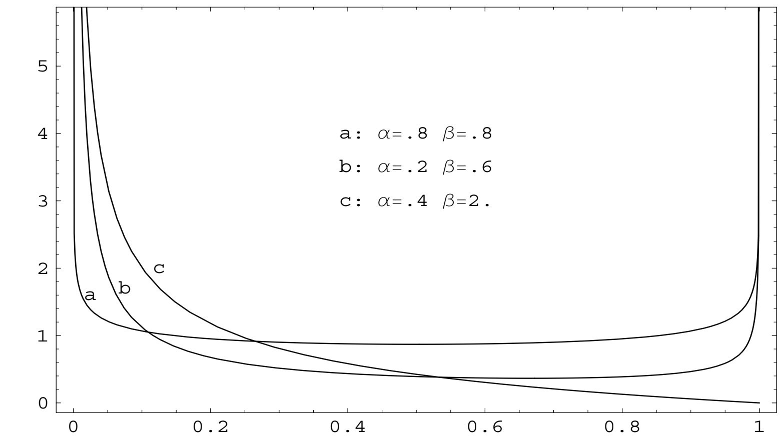


Figure 5: Plots of pdf of beta distribution

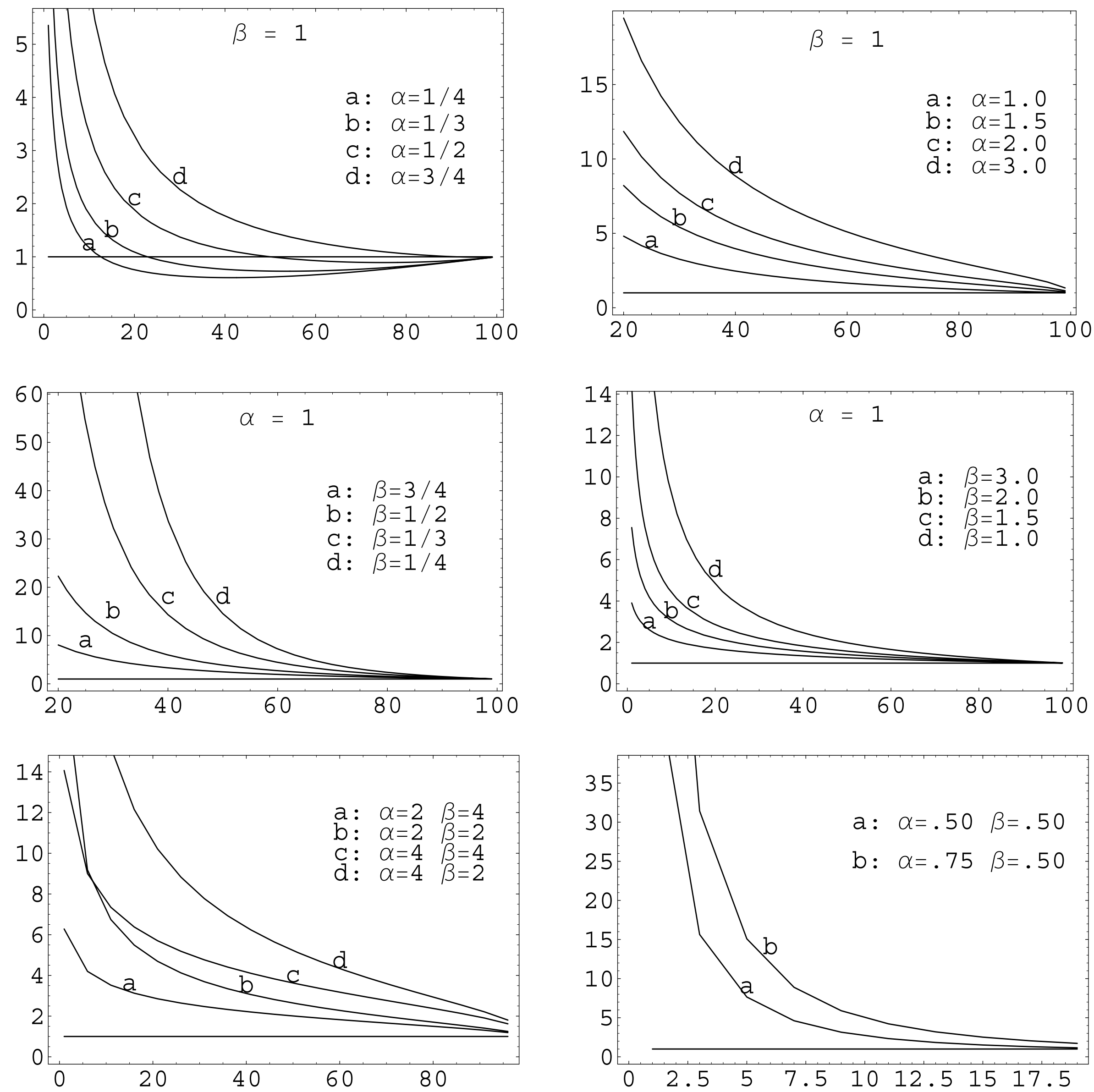


Figure 6: Plots of S^R/S^Q vs m for beta distribution, $n = 100$ for the first five plots and $n = 20$ for the last one. The last two plots are based on a few selected points.

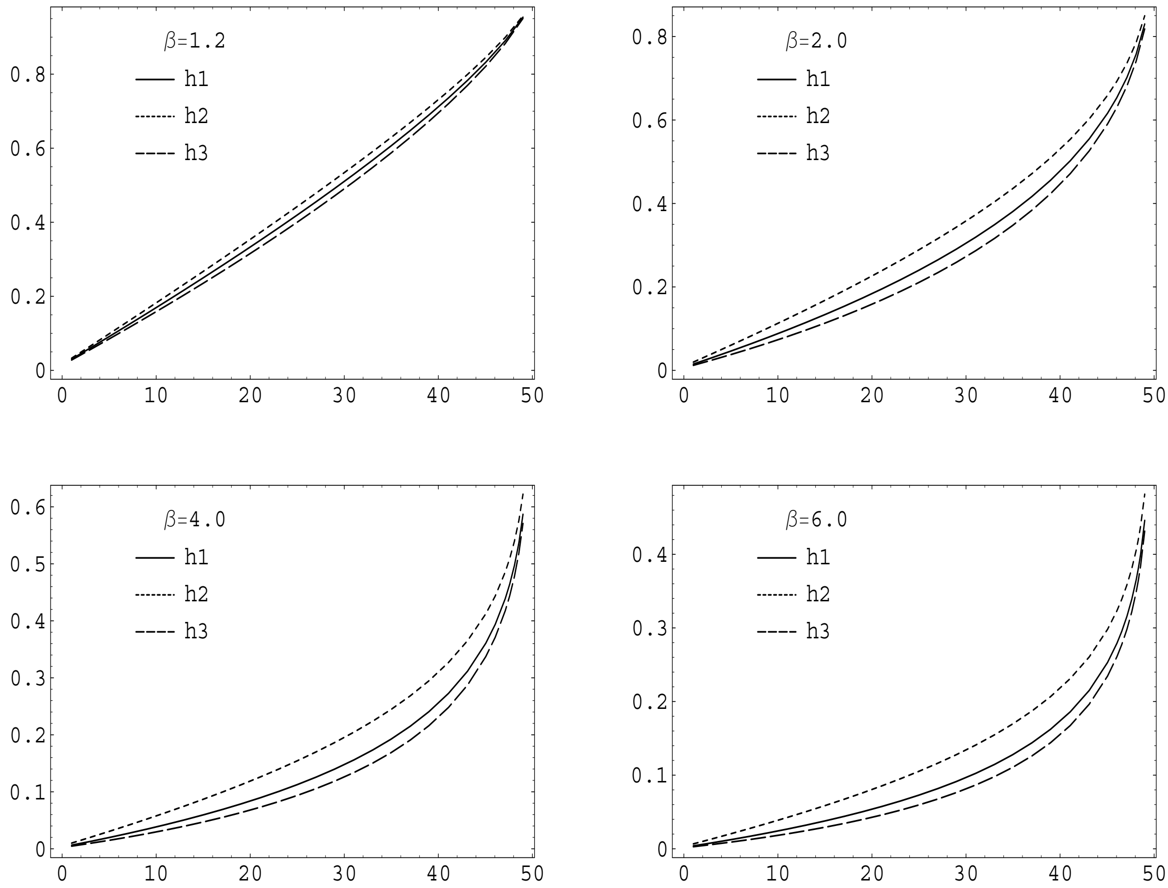


Figure 7: Plots of the three h functions defined in Lemmas 1, 5 and 6: $n = 50$